Lecture 1 :

04-09-18

Motivation : (Newton's 2^{nd} Law)

High school:

$$\boxed{a = F/m}$$

const (on m)

const (on a) $\rightsquigarrow$ $\begin{cases} v(t) = v_0 + at \\ x(t) = x_0 + v_0 t + at^2 \end{cases}$

Calculus :

$$x''(t) = f(t)$$

or

$$\begin{cases} v'(t) = f(t) \\ x'(t) = v(t) \end{cases}$$

Def: An ordinary differential equation is an equation involves derivatives $y^{(0)}(t) = y(t), \ldots, y^{(n)}(t)$

$$F(t, y^{(0)}(t), \ldots, y^{(n)}(t)) = 0$$

- $\underline{n}$ is called the order of the equation.

  We usually refer to the non-trivial order.

Notation:

1. Independent variable : t

2. Interval of definition : $I \subseteq \mathbb{R}$

   $y : I \longrightarrow \mathbb{R}$ is the domain.

3. dependent variable : y

Example: (Caculus)

1. $y' = \cos t$ (1st order ODE)

Solve: $y = \int \cos t = \sin t + C$ $\in \mathbb{R}$

2. $y'' = \cos t$ (2^{nd} order ODE)

Solve:

$$y' = \sin t + C_1$$
$$y = \int (\sin t + C_1)$$
$$= -\cos t + C_1 t + C_2 \quad , \quad C_i \in \mathbb{R}$$

3. $y^{(n)} = 0$ (n-th order ODE)

Solve: $y = a_{n-1} t^{n-1} + \cdots + a_0$, $a_i \in \mathbb{R}$

Observation:

There are infinitely many solutions to an ODE.

Question:

How to describe all of them?

Back to Example:

1. We fix C by prescribing $y(0) = y_0$ (or any $t_0 \in I$)
2. We fix C_1 by prescribing $y'(0)$
   C_2 by — " — $y(0)$
3. We fix a_i's by precribing
   $y^{(n-1)}(0), \ldots\ldots, y(0)$

> **Observation:** # of initial conditions needed is the same as the order of the ODE

Def: An IVP (initial value problem) to an n-th order ODE: $F(t, y^{(0)}, \ldots, y^{(n)}) = 0$

is $t_0, \ldots, t_{n-1} \in I$, $c_0, \ldots\ldots, c_{n-1}$

s.t. $y^{(0)}(t_0) = c_0, \ldots, y^{(n-1)}(t_{n-1}) = c_{n-1}$

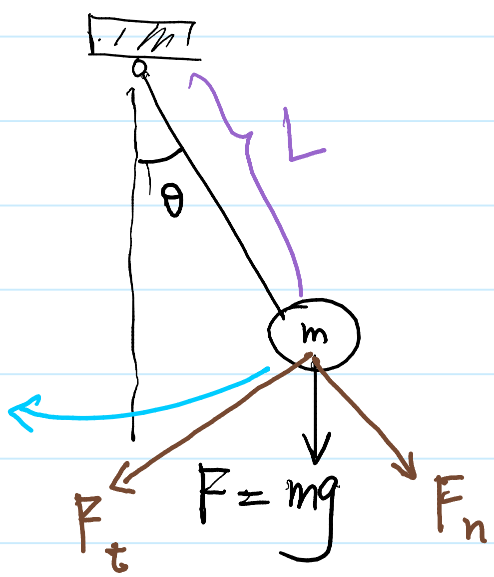
Def: • A <u>Linear</u> ODE if F is Linear in the sense

$$F(t, y^{(0)}, \ldots, y^{(n)}) = a_n(t)\, y^{(n)} + \cdots + a_0(t)\, y^{(0)} - f(t)$$

• A linear ODE is called <u>homogeneous</u> if $f(t) \equiv 0$.

• It is said to have <u>constant coefficient</u> if $a_i(t) \equiv a_i$

Example: (Non-Linear equation)



tangential force : $F_t = mg \sin\theta$

normal force : $F_n = mg \cos\theta$.

Let tangential velocity $= v(t)$.

$\Rightarrow \quad v(t) = L\, \theta'(t)$.

Newton's 2^{nd} law along tangential direction

$$\Rightarrow \boxed{L\theta''(t) = -g\sin\theta}$$

non-linear!

Example (population). $y(t)$: population at time t.

$$\frac{dy}{dt} = \underbrace{h(y)\cdot y}_{\text{growth rate proportional to } y}$$

$h(y)$ need to satisfy:

1) $h(y) > 0$ for y small

(less competition)

2) $h(y) < 0$ for y large enough

$h'(y) < 0$ for y " "

(more competition for house, education etc)

For example: simplest model we can take

$$h(y) = r - ay$$

r: reproduction rate; ay: elimination rate

The ODE become:

$$\underbrace{\frac{dy}{dt} = (r - ay)y}_{\text{non-linear ODE.}}$$

We will learn:

1. (1^{st} order ODE) $y'(t) = f(t,y)$.
(chapter 2)

2. (2^{nd} order linear equation) (chapter 3)

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t).$$

3. (higher order linear equation) (chapter 4)

$$a_n(t)y^{(n)}(t) + \cdots\cdots + a_0(t)y(t) = f(t)$$

4. System of 1^{st} order ODE: (chapter 7)

$$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \underbrace{A(t)}_{\text{n×n matrix of function.}} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

5. Behaviour of solution of (non-linear) 1^{st} order ODE (system) (chapter 9)

6. (If time allow) Fourier series (chapter 10)

7. (If time allow) Prove existence and uniqueness

We may study ODE by:

1. Find an explicity formula for $y(t)$.

   i) in terms of elementary functions
   e^{ct}, $\sin ct$, $\cos ct$ etc.

   (O.K. for linear with constant coefficient and nice $f(t)$)

   ii) in terms of integral formula

2. Abstract Existence and Uniqueness results

Aim: IVP has a unique solution locally for a short period of t.

i) Decribe the behaviour of the solution as t varies.

ii) Do numerical approximation (Not in this course)

1^{st} order equation: $\quad y' = f(t, y)$

constant

Example: IVP $\begin{cases} y' = ay + b \\ y(t_0) = y_0 \end{cases}$

Two cases:

i) $(a = 0)$ $\quad y' = b$

$$\Rightarrow \quad y = bt + c$$

$$y(t_0) = bt_0 + c = y_0$$

$$\Rightarrow \quad c = y_0 - bt_0$$

$$\Rightarrow \quad y = y_0 + b(t - t_0).$$

ii) $(a \neq 0)$

$$y' - ay = b$$

$$(e^{-at} y)' = b \cdot e^{-at}.$$

$$\Rightarrow \quad e^{-at} y = \frac{-b}{a} e^{-at} + c$$

$$y = \frac{-b}{a} + c \cdot e^{at}.$$

$$y(t_0) = y_0$$

$$\Rightarrow \quad y_0 = \frac{-b}{a} + ce^{at_0}$$

$$c = \left(y_0 + \frac{b}{a}\right) e^{-at_0}$$

Hence $$y = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}.$$

Rk: The explicit formula of the solution may depend on the value of coefficients

Ex 2: $y' = p(t)y$, find general sol.

Idea: $$\frac{y'}{y} = p(t)$$

$$(\log|y|)' = p(t)$$

$$\rightsquigarrow \quad \log|y| = \int p(t)\,dt + c$$

$$|y| = k e^{\int p(t)\,dt} \quad \text{where } k = e^c$$

Precisely: $$\left(e^{\int -p(t)\,dt} y(t)\right)' = e^{-\int p(t)\,dt}\left(y'(t) - p(t)y(t)\right)$$

Therefore we can write equation as

$$\left(e^{-\int p(t)dt} y(t)\right)' = 0.$$

$$\Rightarrow \quad y(t) = k e^{\int p(t)\,dt} \quad \text{a general solution}$$

§ Integration factor:

$$\begin{cases} y' = p(t)y + q(t) \\ y(t_0) = y_0 \end{cases}$$

Recall: when $p(t) \equiv a$

we can write $e^{-at}(y' - ay) = (e^{-at}y)'$

and solve the equation.

In general: Want $\mu(t) \neq 0$ for $t \in I$ s.t.

$$\boxed{(\mu(t)\, y(t))' = \mu(t)\,(y'(t) - p(t)\,y(t))}$$

Such $\mu(t)$ is call integrating factor.

Solving for integrating factor:

Equation: $\left(u'(t) + p(t)u(t)\right) y(t) = 0$

We want: $u'(t) + p(t)u(t) = 0$

solve: $u(t) = k e^{-\int p(t)dt}$

With an integration factor:

$$y' - py = q$$

(multiple by u) $\mu y' - \mu p y = q\mu$

$$(\mu y)' = q\mu$$

$$\Rightarrow \mu y = \int q\mu \, dt + c$$

$$\Longrightarrow y = \frac{1}{\mu}\left[\int q u \, dt + c\right]$$

Remark: For $\mu = k e^{-\int p dt}$

- there is an freedom to choose $k \neq 0$.
- Observe that in the result for y, we have

$$y = \frac{1}{\mu}\left[\int q \mu dt + c\right]$$

and the effect of k cancel

Choose $k=1$:

$$\boxed{y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt} g(t)\,dt + c\right]}$$

Example : $ty' + 2y = 4t^2$

Step 1 : write it into standard form

$$y'(t) = p(t)y(t) + g(t) \quad :$$

$$y' = \frac{-2}{t} y + 4t = g(t).$$

$p(t)$

Notice : $p(t)$ Not define for $t=0$

Step 2 : Solve for integrating factor

$$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\log|t|} = t^2$$

Step 3 : plug into the formula

$$y(t) = \frac{1}{t^2}\left[\int t^2 \cdot 4t\,dt + c\right]$$

$$= t^2 + \frac{c}{t^2}.$$

Rk: For $C \neq 0$, $y(t)$ is <u>NOT</u> defined at $t \neq 0$

- So we can only talk about the equation on $t \in (-\infty, 0) \cup (0, +\infty)$

- As we will see from the uniqueness theorem IVP with $t_0 \in (0, +\infty)$ will determine the solution on $(0, +\infty)$.

- So we will talk about the equation on either one of the interval.